Chapter 6: Fundamental Methods Used In Structural Dynamics

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6.1 Introduction:

Structural dynamics is a fascinating subject. After you get into it and understand it, you will most likely become "hooked". On the flip side, if you don't understand the fundamentals, you will quickly get lost. There are layers of complexity to this subject, which need to be appreciated at each level of understanding. It will take some effort to get the big picture.

The intent of this chapter is to take a broad view of this subject, summarizing in one short paper the fundamental solutions. A hierarchy of methods used in structural dynamic problems will be presented, with step-by-step procedures that can be applied in spreadsheets and software applications.

Several types of dynamic problems that structural engineers typically encounter:

- Machine foundations (foundations subject to out-of-balance machines)
- Beam vibration analysis (floors subject to walking/dancing)
- Smoke stacks, signs, cables and bridges subject to wind vortex shedding forces
- Seismic design (buildings/bridges subject to earthquakes)

At first glance, these design situations appear to be quite different. Actually the only major difference is the nature of the force or support movement. After defining forces or support movements, all of these problems can be analyzed using similar methods.

> One year, I was involved with the design of two projects requiring dynamic analysis: A building housing four 3600 H.P. generators, requiring foundation designs; and a large plaza structure to be designed for a 0.6g seismic event.
>
> Let me clarify that I'm from northern Illinois, where we don't get earthquakes; so, the latter project was quite an intimidating challenge! It was expedient that I brush-up my knowledge of structural dynamics and get a good grasp of the subject. I certainly had a lot of questions.
>
> If you are new to this subject, you will probably be asking many of the same questions I asked. I firmly believe that it is important to understand the essential mathematics to gain confidence in this subject. I have added historical comments about the inventors of these methods, hoping to make things more interesting and memorable. A list of references is presented at the end of this chapter.

* DesignCalcs, Inc. – Structural Engineering Software (www.dcalc.us)

6.2 An Overview of Structural Dynamics:

The subject of structural dynamics can be broken into the analysis of two types of systems:

1. Single Degree of Freedom (SDOF) Systems:

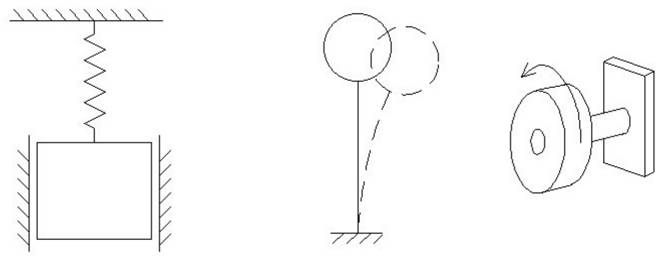


Fig. 1: Examples of Single Degree of Freedom Systems

Single degree of freedom systems are comprised of a mass (m) connected to a spring, beam or shaft having a stiffness (k). The above sketch represents three schematic single degree of freedom systems. The movement of each of these three systems can be described using similar structural dynamic concepts.

2. Multiple Degree of Freedom (MDOF) Systems:

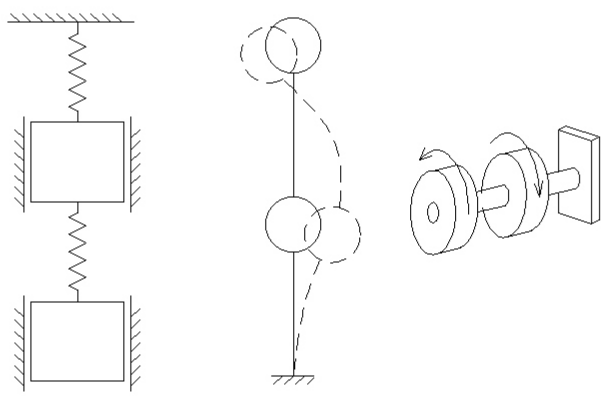


Fig. 2: Examples of Multiple Degree of Freedom Systems

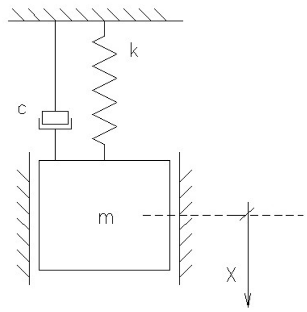
Multiple degree of freedom systems are comprised of several discrete masses, or continuous masses. A multi-story building is a classic example of a system having discrete masses at each story. A beam is a good example of a system that has a continuously distributed mass.

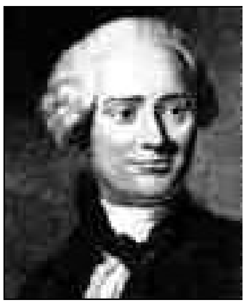
There are two types of excitations that can be applied to SDOF and MDOF systems,

- Excitations caused by a force:
  - Sinusoidal force (example: out-of-balance machine)
  - Non-sinusoidal force (example: a person walking on a beam)
- Excitations caused by a support displacement:
  - Sinusoidal displacement
  - Non-sinusoidal displacements (example: an earthquake)

6.3 Why Do Structures Vibrate?:

All structures vibrate to some degree. This is something we take for granted. But why exactly *do* structures vibrate? The explanation comes from a differential equation that is the basis of structural dynamics.





In the 1700's, Jean Le Rond d'Alembert developed a principle that states that inertial forces in mechanical systems can be treated in the same way as static forces in static systems.

Looking at the above basic system, there are three forces:

inertia force = mx''
viscous damping force = cx'
spring force = kx

where

x' = first derivative of x with respect to time, t
x''= second derivative of x with respect to time, t

Applying d'Alembert's principle we check for force equilibrium as follows:

$$mx'' + cx' + kx = 0$$

Trying $x = e^{st}$ results in the following,

$$ms^2e^{st} + cse^{st} + ke^{st} = 0$$
$$(ms^2 + cs + k)e^{st} = 0$$

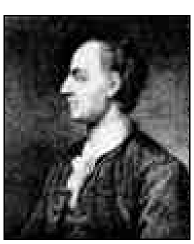
Since "e^{st}" is not zero, the term "$(ms^2 + cs + k)$" must be zero. Therefore,

$$ms^2 + cs + k = 0$$

We solve for "s" using the quadratic equation,

$$s = -(c/2m) \pm \sqrt{(c/2m)^2 - k/m}$$

If the radical term "$(c/2m)^2 - k/m$" is negative, "s" is a complex number, "$\alpha + j\beta$".



Leonhard Euler (who incidentally knew d'Alembert) discovered the relationship, $e^{j\beta t} = \cos \beta t + j \sin \beta t$. We will utilize Euler's equation here to draw the following conclusions:

- When "s" is a complex number, a system will respond with sinusoidal vibrations.
- When "s" is not complex, a system will not vibrate

6.4 Free Motion of System Without Damping

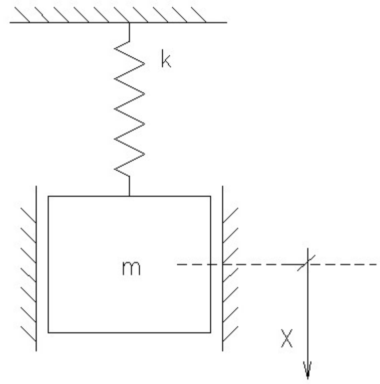


Fig. 3: System Without Damping

Differential Equation:

$$mx'' + kx = 0$$

where,

m = mass (weight in pounds divided by g=32.2 ft/sec^2 or 386 in/sec^2)
k = spring constant (pounds per inch or feet)
x = displacement (feet or inch) at time, t (seconds)
x'' = second derivative of x with respect to time

Solution: (Ref. 1, p. 31)

$$x = c_1 \sin \omega_n t + c_2 \cos \omega_n t$$

where,

ω_n = the natural frequency = $\sqrt{k/m}$ (radians per second)

The constants "c_1" and "c_2" are determined based on initial boundary conditions.

An alternative expression for "x" can be determined as follows:
At time, t=0, the **initial displacement** of the system is

$\mathbf{x_0} = c_1 \sin(0) + c_2 \cos(0) = c_2$

The velocity of the system at any time is,

$x' = c_1\omega_n \cos \omega_n t - c_2\, \omega_n \sin \omega_n t$

and the **initial velocity** is,

$\mathbf{v_0} = c_1\omega_n \cos(0) - c_2\, \omega_n \sin(0) = c_1\omega_n$. Therefore $c_1 = v_0/w_n$

The expression for "x" is sometimes written in the following form:

$$x = (v_0/w_n)\sin \omega_n t + x_0 \cos \omega_n t$$ (Ref. 2, p. 13)

6.5 Free Motion of System With Damping:

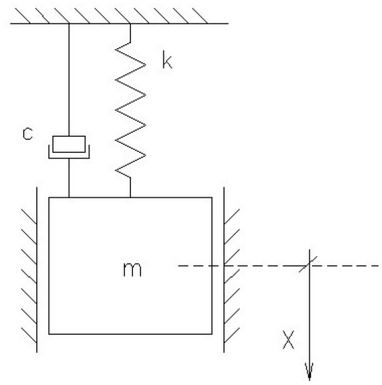


Fig. 4: System With Damping

Differential Equation:

$$mx'' + cx' + kx = 0$$

where,

m = mass (weight in pounds divided by g=32.2 ft/sec^2 or 386 in/sec^2)
c = viscous damping coefficient
k = spring constant (pounds per inch or feet)
x = displacement (feet or inch) at time, t (seconds)
x' = derivative of x with respect to time, t
x'' = second derivative of x with respect to time

Solution: (Ref. 1, p. 37)

Critical damping coefficient, $c_c = 2\,m\,\omega_n$
ω_n = the natural frequency = $(k/m)^{1/2}$

Case 1: $c > c_c$
$x = c_1 e^{s1t} + c_2\, e^{s2t}$
where
$s_1 = -(c/2m) + [(c/2m)^2 - k/m]^{1/2}$
$s_2 = -(c/2m) - [(c/2m)^2 - k/m]^{1/2}$

Case 2: $c < c_c$
$x = e^{-(c/2m)t}\,(c_1 \cos qt + c_2 \sin qt)$
where
q = damped natural frequency = $[(k/m) - (c/2m)^2]^{1/2}$ (rad./sec)

Typically "c" is not known. In practice we use a damping ratio,
ε = damping ratio = c/c_c
A common value of "ε" used for structures is 5% critical damping.

6.6 Single Degree of Freedom System Subject to Sinusoidal Force:

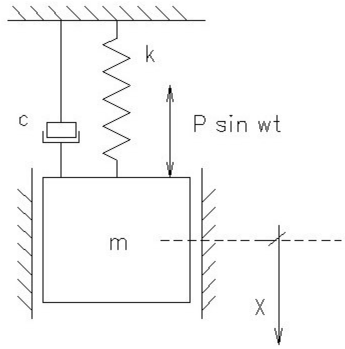


Fig. 5: System Subject To A Sinusoidal Force

Differential Equation:

$$mx'' + cx' + kx = P \sin \omega t$$

where,

m = mass (weight in pounds divided by g=32.2 ft/sec^2 or 386 in/sec^2)
c = viscous damping coefficient
k = spring constant (pounds per inch or feet)
P = maximum amplitude of force (pounds or kips)
ω = angular rate of change of forcing function (radians per second)
t = time (seconds)
x = displacement (feet or inch) at time, t
x' = derivative of x with respect to time, t
x'' = second derivative of x with respect to time

Solution: (Ref. 1, p. 47)

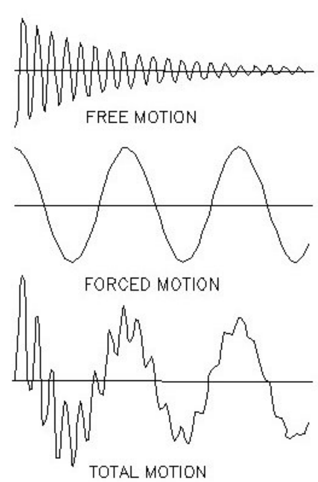
x = free motion + forced motion

The "free motion" part of the solution is described in section 6.5

The "forced motion" part of the solution is,
$x = x_0 \sin (\omega t - \phi)$

where
$x_0 = (P/k)/[(1 - (\omega/\omega_n)^2)^2 + (2c/c_c * \omega/\omega_n)^2]^{1/2}$

$\tan \phi = 2c/c_c * \omega/\omega_n/[1 - (\omega/\omega_n)^2]$



6.7 Single Degree of Freedom System With Sinusoidal Support Movement:

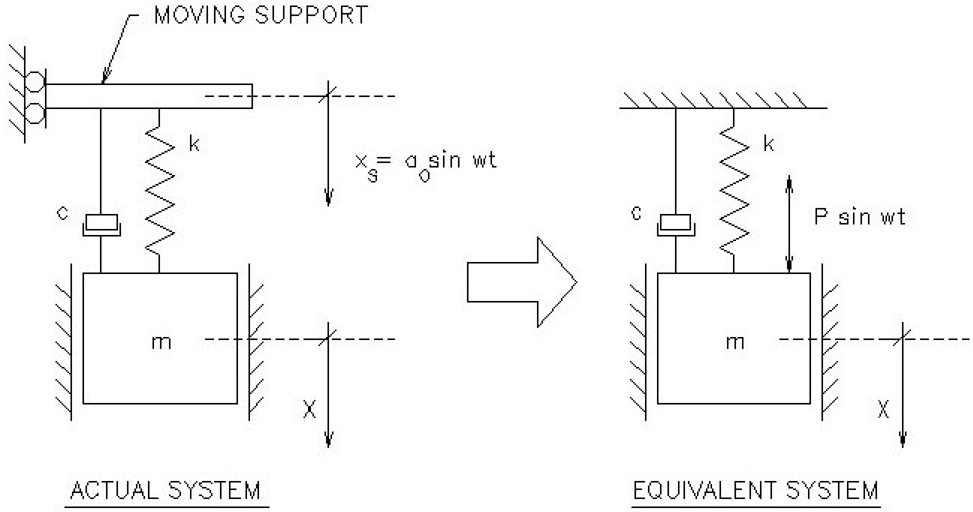


Fig. 6: System Subject to Sinusoidal Support Movement

Differential Equation:

$$mx'' + k(x - a_0 \sin\omega t) + c(x' - a_0\omega \cos\omega t) = 0$$

or rearranging,

$$mx'' + cx' + kx = ka_0\sin\omega t + c\, a_0\omega\cos\omega t$$
$$= [(ka_0)^2 + (c\, a_0\, \omega)^2]^{1/2} \sin(\omega t + \phi)$$

We recognize that the right hand side has the same form as a sinusoidal force.

Solution: (Ref. 1, p. 30)

A sinusoidal support movement is equivalent to a sinusoidal force applied at the mass,

$$P(t) = [(ka_0)^2 + (c\, a_0\, \omega)^2]^{1/2} \sin(\omega t + \phi)$$

For practical applications, we ignore the phase shift "ϕ" and apply a force,

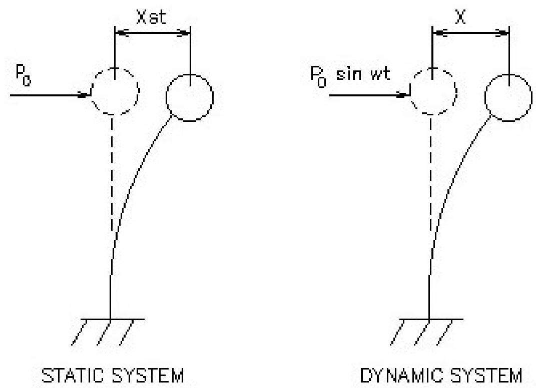
$$P(t) = [(ka_0)^2 + (c\, a_0\, \omega)^2]^{1/2} \sin(\omega t)$$

to the equivalent system.

Using this substitution, problems like these can be solved using the method described in section 6.6.

6.8 Transmissibility Factor:

Section 6.6 presented the most important solution for real world problems: The response of a damped system to a forcing function. The main question structural engineers must ask is, what effects does a dynamic force have on a structure?

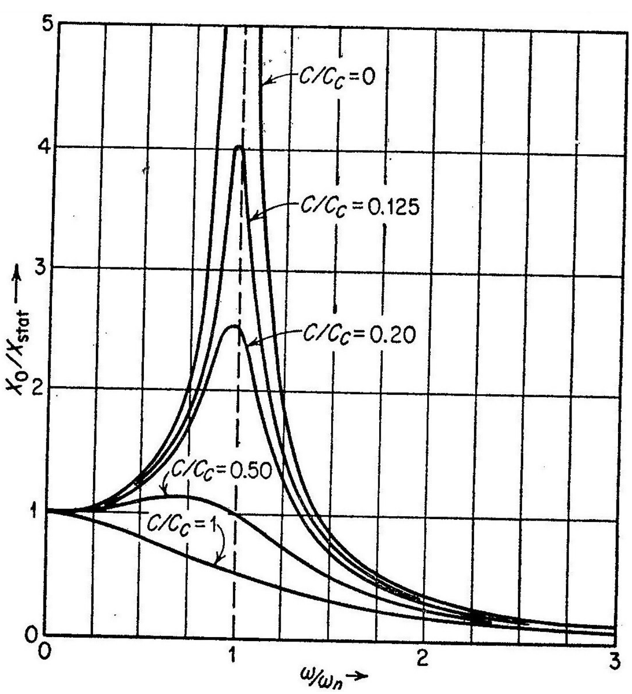


If "x_{st}" is the response of a system to a force "P_0", and if "x" is the maximum response of a force to a dynamic force, "$P_0 \sin \omega t$", transmissibility is defined as,

$$\text{Transmissibility} = x/x_{st}$$

Using the previous analysis results, transmissibility can be computed by,

$$\text{Transmissibility} = \sqrt{\frac{1 + [2\,c\,\omega/(c_c\,\omega_n)]^2}{[1 - (\omega/\omega_n)^2] + [2\,c\,\omega/(c_c\,\omega_n)]^2}} \qquad \text{(Ref. 1, p. 71)}$$



Transmissibility Factors (Ref. 1, p. 51)

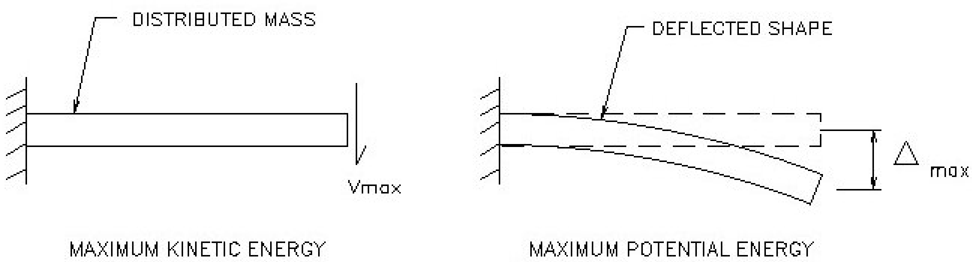
This graph can be quite useful, allowing a quick and easy evaluation of some problems.

6.9 Solving For The Natural Frequency Of Beams With Distributed Masses:

The natural frequency of systems with distributed masses cannot be solved simply from a differential equation. For complex systems such as beams, there is an alternative approach for computing natural frequency, invented by Lord Rayleigh in the 1800's.

This method is very straight forward and has the advantage of being intuitively simple.

To illustrate the method, the cantilever beam shown below is a system with a distributed mass:



Energy conservation requires that the total energy of this closed system must remain constant. Therefore

K.E. (max) = P.E. (max)

Rayleigh's Method:
(Ref. 1, p. 141)

1. Assume a force, $q(x) = 1$ per unit length, is applied to the beam
2. Compute the deflected shape, $a_m(x)$, using conventional methods
3. Assume sinusoidal dynamic behavior: $a(x) = a_m(x) \sin \omega t$
4. The velocity is computed by differentiating $a(x)$,
   $v(x) = \omega\, a_m(x) \cos \omega t$
5. Similarly, the acceleration is computed by differentiating $v(x)$,
   $a(x) = -\omega^2\, a_m(x) \sin \omega t$
6. The maximum velocity at any point is,
   $v_{max}(x) = \omega\, a_m(x)$
7. Compute total K.E. by integrating along the length of the beam:
   $K.E.\ max = \Sigma\ 1/2\ [\omega\, a_m(x)]^2\, m(x)\, \Delta x$ (summing small segments)
8. Compute the total P.E. by integrating along the length of the beam:
   $P.E.\ max = \Sigma\ 1/2\ q(x)\, a_m(x)\, \Delta x$
9. Setting K.E. max = P.E. max, solve for ω

The solution for "ω" can be further refined by the following iteration:

10. Compute maximum acceleration,
    $a_{max}(x) = \omega^2\, a_m(x)$
11. Revise the force,
    $q(x) = \omega^2\, a_m(x)\, m(x)$
12. Go to step 2, iterating the solution

6.10 Single Degree of Freedom System Subject to A General Force:

There are many situations where structures are subject to non-sinusoidal forces.

- Structures subject to explosions.
- Beams subject to walking
- Structures subject to seismic support displacements

In these cases, the previous methods based on sinusoidal forces cannot be used, and numerical techniques are necessary.

One of the most useful concepts for the analysis of single degree of freedom systems was invented by Jean Marie Constant Duhamel in the 1800's.

This concept, called "Duhamel's Integral"* , is an elegant mathematical concept. It will be explained in detail because of its importance.

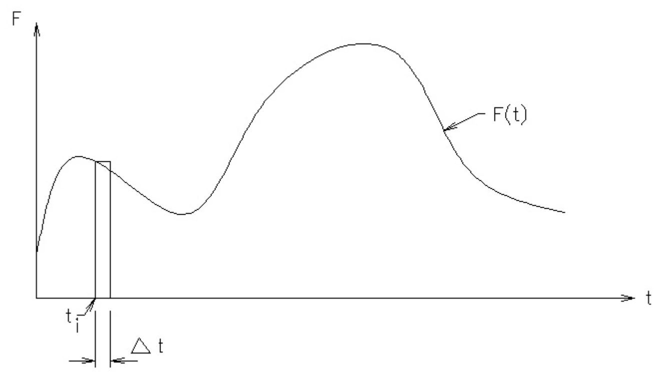


Fig 7: A General (Non-Sinusoidal) Force

A constant force, "F_i", acting on a mass will cause a constant acceleration of the mass,

$$a = F_i/m$$

For a mass that is initially at rest, the velocity of the mass after a short time "Δt" will be,

$$v_i = a\,(\Delta t) = (F_i/m)\,(\Delta t)$$

Now, let's assume that the mass is part of a single degree of freedom system without damping. If the time duration "Δt" is chosen to be very short, and if the spring has no initial displacement, then the initial velocity of the mass will be "v_i".We will temporarily rename this velocity "v_0", for reasons that will become apparent.

The dynamic behavior of a single degree of freedom system (covered in section 6.3) is,

$$x = (v_0/\omega_n) \sin \omega_n t + x_0 \cos \omega_n t$$

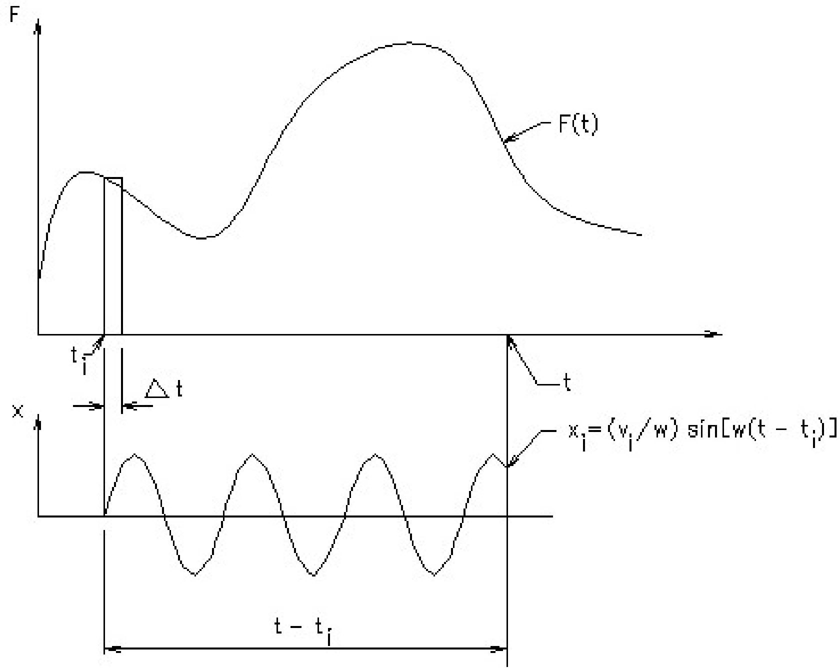
For a system without initial displacement, this can also be rewritten in a form,

$$x = (v_o/\omega_n) \sin \omega_n t,$$

where "v_o" is the initial velocity of the mass of the system.

(* See Reference 2, Chapter 4)

6.10 Single Degree of Freedom System Subject to a General Force (Cont'd):



Looking at a short burst of force "F_i" applied to one system at time "t_i", the velocity is

$$x_i = (v_i/\omega_n) \sin \omega_n(t-t_i) = (F_i/m)\, \Delta t/\, \omega_n \sin \omega_n(t-t_i)$$

Proceeding along this line of thought, we can divide the force into separate bursts applied to an identical system but at a different time "t_i". The displacement "x_i" is computed for each of these identical systems.

The total displacement "x" is the superposition of the bursts of force applied to every identical system.

$$x_{\text{total}} = \Sigma\, (F_i/m\omega_n)\, \Delta t \sin \omega_n(t-t_i)$$

The above is "**Duhamel's Integral**" **for a single degree of freedom system without damping**, written as a summation for numerical integration.

For damped systems, a similar approach is used, resulting in **Duhamel's Integral for damped systems**,

$$x_{\text{total}} = \Sigma\, e^{-\varepsilon\omega(t-ti)}(F_i/m\omega_D)\, \Delta t \sin \omega_D(t-t_i)$$

where

$$\varepsilon = \text{damping ratio} = c/c_{cr}$$

The above two integrals are somewhat complex, because of the term "$\sin \omega(t-t_i)$". In practice, for numerical evaluation, alternative forms of these equations are used.

6.11 Single Degree of Freedom System Subject to Support Acceleration:

Pertinent to seismic analysis, we need to be able to compute the excitation of a structure to general (non-sinusoidal) support displacements. The method developed by Duhamel described in the previous section can easily be extended for this purpose.

In the 1950's, Professor Nathan Newmark of the University of Illinois pioneered many of the methods that are used today for seismic design. His work formed the foundation of the present day seismic design codes.

One of the first things he determined was how to calculate the response of single degree of freedom systems to recorded seismic data.

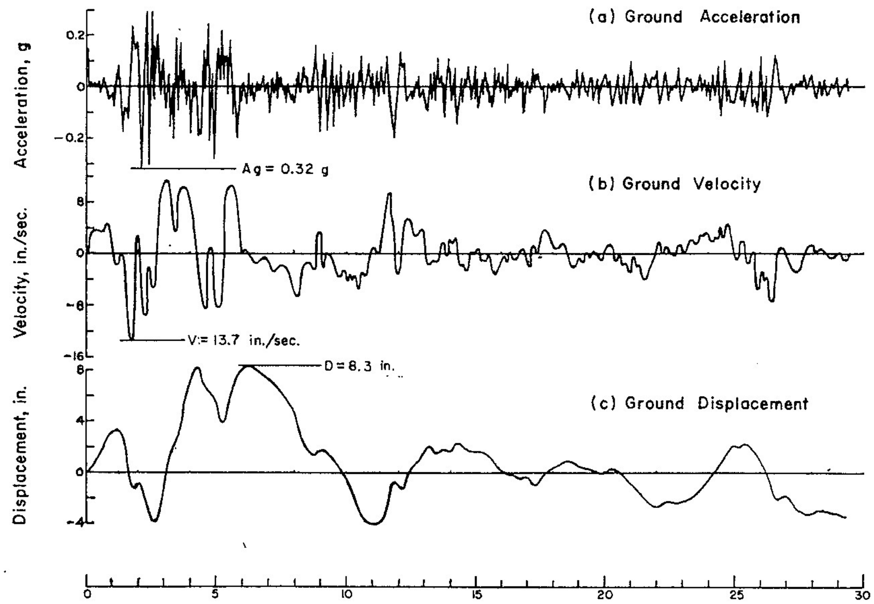
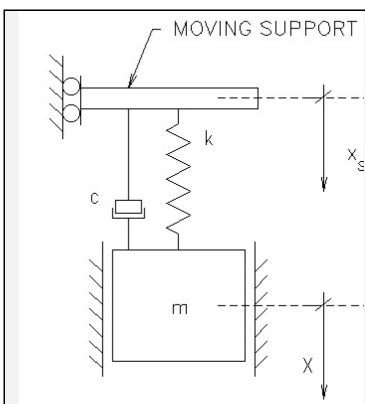


Fig. 8: El Centro, California, earthquake of May, 18, 1940

Records of the 1940 El Centro California earthquake where sufficiently detailed for numerical analysis. Newmark used this data to compute the response of various SDOF systems to ground displacements caused by this earthquake and others.



Response of undamped single degree of freedom system to general support motion, relative to support:

$$u = x - x_s = \Sigma\, (-1/\omega_n) x_s'' \, \Delta t \, \sin\omega_n(t-t_i)$$

Response of damped single degree of freedom system to general support motion, relative to support:

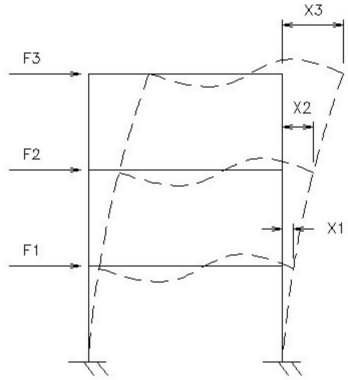
$$u = x - x_s = \Sigma\ e^{-\varepsilon\omega(t-ti)} (-1/\omega_D) x_s'' \, \Delta t \, \sin\omega_D(t-t_i)$$

where x_s'' = support (ground) acceleration.

(Ref. 3)

6.12 Natural Periods of Multiple Degree of Freedom System: (Ref. 2, Ch. 10)

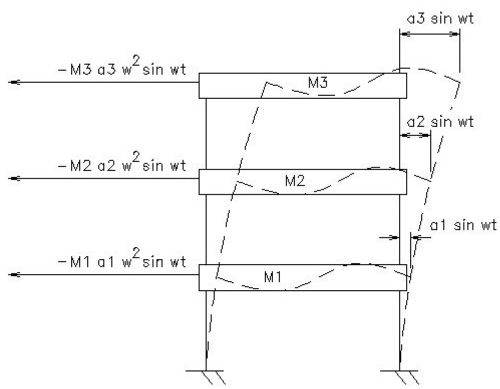
In practice, rarely do we encounter single degree of freedom systems. We are typically designing buildings and bridges that have multiple degrees of freedom.



Complex structures are typically analyzed using matrix stiffness analysis. For example, a three degree of freedom structure may have the following stiffness equation,

$$\begin{vmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} F_1 \\ F_2 \\ F_3 \end{vmatrix} \quad \text{or,} \quad K * X = F \text{ written in matrix notation.}$$

Complex structures without damping will vibrate in several natural frequencies, "ω". For each natural frequency, "ω", there is an associated "normal mode shape".



For sinusoidal movement, the deflected shape at each non-zero mass node is,

$$x_i = a_i \sin \omega t$$

and the acceleration of each degree of freedom is,

$$x_i'' = -a_i \, \omega^2 \sin \omega t$$

If we ignore the rotational degrees of freedom and consider only translations, the inertia of each mass is,

$$m_i x_i'' = -m_i \, a_i \, \omega^2 \sin \omega t$$

Dynamic equilibrium of each mass requires, mx'' + kx = 0. Written in matrix form,

$$\begin{vmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{vmatrix} \begin{vmatrix} -a_1 \omega^2 \sin \omega t \\ -a_2 \omega^2 \sin \omega t \\ -a_3 \omega^2 \sin \omega t \end{vmatrix} + \begin{vmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{vmatrix} \begin{vmatrix} a_1 \sin \omega t \\ a_2 \sin \omega t \\ a_3 \sin \omega t \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}$$

6.12 Natural Periods of Multiple Degree of Freedom System (Cont'd):

Eliminating the "sinωt" and combining the square matrices results in,

$$\begin{vmatrix} k_{11} - m_1\omega^2 & k_{12} & k_{13} \\ k_{21} & k_{22} - m_2\omega^2 & k_{23} \\ k_{31} & k_{32} & k_{33} - m_3\omega^2 \end{vmatrix} \begin{vmatrix} a_1 \\ a_2 \\ a_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}$$

The above equation written in matrix notation is,

$$[K - M\ \omega^2\ ] * A = 0$$

Since the "A" vector can assume any value, we conclude that

$$\text{Determinant } [K - M\ \omega^2\ ] = 0$$

This is called an **Eigenvalue problem**. Problems of this type are also encountered in advanced physics and mathematics. (Vibration analysis is historically a problem in classical physics.)

To illustrate the difficulty of the solution, we continue with our little three degree of freedom problem. The evaluation of the determinant for the three degree of freedom problem will be a cubic equation of the form,

$$c_0 + c_1\ (\omega^2) + c_2\ (\omega^2)^2 + c_3\ (\omega^2)^3 = 0$$

This cubic equation has three roots for "ω^2", which can easily be solved. Then, "ω" is the square root of "ω^2". Therefore, a system having three masses, each having one direction of freedom, has three possible modes of vibration.

For a system with "n" masses, each having one direction of freedom, evaluation of the determinant will result in a polynomial of the form,

$$c_0 + c_1\ (\omega^2) + c_2\ (\omega^2)^2 + c_3\ (\omega^2)^3 + \ldots + c_n\ (\omega^2)^n = 0$$

A polynomial of order "n" will have "n" roots. We conclude then,

> A structure with "N" non-zero mass nodes, each having "M" directions of freedom will have "N*M" possible normal modes of vibration.

The only way to solve immense problems of this type is with a computer. Several numerical methods have been developed for solving eigenvalue problems. Reference 2 describes computer code developed by Bathe and Wilson in the 1970's.

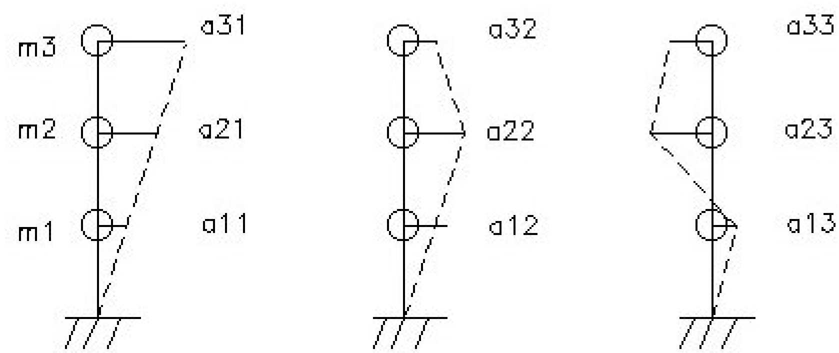
Having determined each vibration mode, "ω", we then ask what is the solution for the "A" vector? In mathematical terms, the "A" vector is an **eigenvector**. However, because the determinant of "$[K - M\ \omega^2\ ]$" is equal to zero, "A" has no single solution! If the determinant of "$[K - M\ \omega^2\ ]$" is equal to zero, this means that two rows in the matrix "$[K - M\ \omega^2\ ]$" are multiples of each other. This is analogous to the problem of finding the intersection of two parallel lines: The lines don't intersect.

6.12 Natural Periods of Multiple Degree of Freedom System (Cont'd):

The conclusion is, **there isn't a *unique* "A" vector solution**. There are actually an infinite number of solutions for "A" that will satisfy the eigenvalue problem. But for a particular solution for "A", we would need initial boundary conditions.

> Each natural period of vibration, "ω", will cause all of the masses to satisfy dynamic equilibrium. Dynamic equilibrium of a structure is only satisfied for deflections sets conforming to certain deflection sets called "modal shapes". The amplitude of a modal shape depends on initial conditions.

To compute the "A" vector, it is necessary to substitute "ω" into the equation "$[K – M\ \omega^2\ ] * A = 0$", and then solve for relative ratios of the "a" dimensions. Typically, one of the "a" dimensions is set equal to unity. The "A" vector for the reduced problem can then be solved, keeping in mind that it is only a proportional "shape".



Shape for ω_1 Shape for ω_2 Shape for ω_3

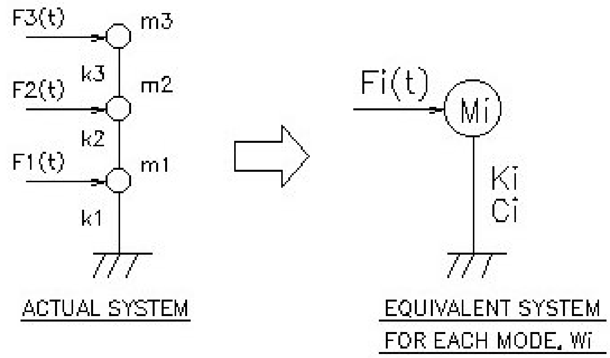
The above figures illustrate conceptually the normal mode shapes for a three degree of freedom system.

> In practice, shape coefficients are normalized using the following equation,
> $$\phi_{ij} = a_{ij}/(\Sigma\ m_i\ a_{ij}^2)^{1/2}$$

Recalling Rayleigh's method, described in section 6.9, we also discussed a shape function, "a(x)". This is no coincidence, because we are talking about the same thing: The shape a structure takes when it is vibrating.

Both Rayleigh's method and the eigenvalue approach are essentially solving the same problem. Rayleigh's method is typically limited to solving the vibration of beams, requiring the designer to make some assumptions about the modal shape. Eigenvalue solutions will reveal many more shapes that may not be intuitively obvious.

6.13 Multiple Degree of Freedom System Subject To A Force: (Ref. 2, Ch 11)
We now have the key ingredients for making a **modal analysis** of the system:

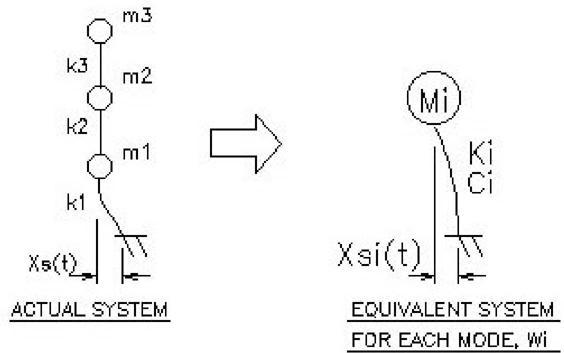


Modal Analysis of a System Subject To A Force:

1. Determine the forces that will be applied the system:
   - Assign force to each mass node, "i". Use the terminology "F_i".
   - Determine how the force varies with time at each node:
     a. Sinusoidal forces: $F_i(t) = F_{io} \sin \omega_f t$ for each nodes, "i".
     b. General forces: Make a table, indicating the force "$F_i(t)$", for each nodes, "i".
2. Determine the natural frequencies and modal shapes:
   - Use a computer program that solves for modal frequencies, "ω_j", and normalized shape coefficients, "ϕ_{ij}" (see section 6.12)
   - Alternatively, Rayleigh's method (see section 6.9) can be used to determine frequencies and shapes (typically only one frequency and shape is computed using this method)
3. For each natural frequency, "ω_j", compute the following:
   - Modal mass: $M_j = \Sigma m_i \phi_{ij}^2$ (result will be M=1)
   - Modal stiffness: $K_j = \omega_j^2 M_i$
   - Modal damping: $C_j = 2\varepsilon_j \omega_j$ (where $\varepsilon_j = c_i / c_{i,cr}$)
   - Modal force: $F_j = \Sigma \phi_{ij} F_i$
4. For each natural mode of vibration, "ω_j", compute the single degree of freedom response, "x_j":
   - Sinusoidal forces: use the method described in section 6.6
   - General forces: use Duhamel's Integral described in section 6.10
5. Based on the results, identify the maximum single degree of freedom response, "x_{jmax}"
6. Compute the maximum responses at each node using the formula:

$$x_{i\,max} = (\Sigma (\phi_{ij} x_{i\,max})^2)^{1/2}$$

6.14 Multiple Degree of Freedom System Subject To Support Movement:



Modal Analysis of System Subject To Support Movement: (Ref. 2, Ch. 11)

1. Determine the support acceleration to be applied to the system:
Make a table of support acceleration (x_s''), velocity (x_s'), or movement (x_s). (Note: Seismic data records of acceleration commonly have "noise" spikes, that can cause errors. Seismologists typically correct the velocity and movement data to remove noise, making this data more useful.)
   - Compute x_s'' by the following approximation,
     a. $x_s'(t)=[x_s(t+\Delta t) - x_s(t)]/\Delta t$
     b. $x_s''(t)=[x_s'(t+\Delta t) - x_s'(t)]/\Delta t$

2. Determine the natural frequencies and modal shapes:
   - Use a computer program that solves for modal frequencies, "ω_j", and normalized shape coefficients, "ϕ_{ij}" (see section 6.12)
   - Alternatively, Rayleigh's method (see section 6.9) can be used to determine frequencies and shapes

3. For each natural frequency, "ω_j", compute the following:
   - Modal mass: $M_j = \Sigma\, m_i\, \phi_{ij}^2$ (result will be M=1)
   - Modal stiffness: $K_j = \omega_j^2 M_i$
   - Modal damping: $C_j = 2\varepsilon_i\, \omega_i$ (where $\varepsilon_i=c_i/c_{i,\,cr}$)
   - Mass participation, $\Gamma_j = \left(\sum_{k=1}^{n} m_k a_{kj}\right) \div \left(\sum_{k=1}^{n} m_k \left(a_{kj}\right)^2\right)$
   - Modal Force: $F_j = -\Gamma_j\, x_s''$ (force applied to mass)

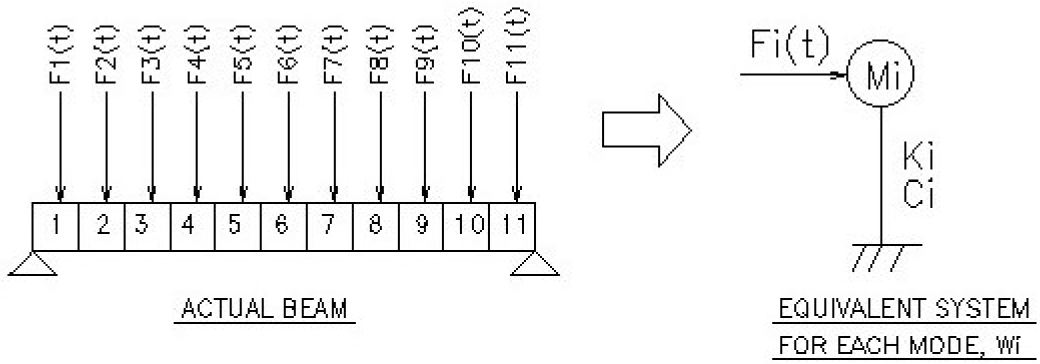
4. For each natural mode of vibration, "ω_j", compute the single degree of freedom response relative to support, "u_j", for a system subject to F_j.

5. Identify the maximum single degree of freedom response "$u_{j,max}$".

6. The maximum response at each node is typically computed as,

$$u_{i,\,max} = (\Sigma(\phi_{ij}\, u_{j\,max})^2)^{1/2}$$

6.15 Response of a Beam to a Force:



Response of a Beam to a Force:

1. Make a table. Divide the beam into N short segments, each Δx long. Identify each segment, using indices "i" from 1 to N. Compute the mass, "m_i", of each of the segments.
2. Using Rayleigh's method, determine the beam's main natural frequency, "ω", and modal shape "a_i" at each segment (see section 6.9).
3. Normalize the shape function for each segment,
   $\phi_i = a_i/[\Sigma\, m_i\, a_i^2]^{1/2}$
4. Determine the forces to be applied to the beam at each segment:
   a. Sinusoidal forces: $f_i(t)= f_{io} \sin \omega_f t$, where "$f_{io}$" is the maximum force at a segment
   b. General forces: Make a table, indicating force $f_i(t)$ for each segment at all times "t".
5. Using the natural frequency, "ω", compute the following:
   - Modal mass: $M = \Sigma\, m_i\, \phi_i^2$ (result will be M=1)
   - Modal stiffness: $K = \omega^2 M$
   - Modal damping: $C_j = 2\varepsilon_i\, \omega_i$ (where $\varepsilon_i = c_i/c_{i,\,cr}$)
   - Modal force: $F(t) = \Sigma\, \phi_i\, f_i(t)$
6. Compute the single degree of freedom response, "x_{sdof}".
   - Sinusoidal forces: use the method described in section 6.6
   - General forces: use Duhamel's Integral described in section 6.10
7. Compute the deflection at all points along the beam using

$$x_i = \phi_i * x_{sdof}$$

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