Chapter 14: Non-Linear Structural Analysis Using DCALC

By Karl Hanson, S.E., P.E.*
December 2014

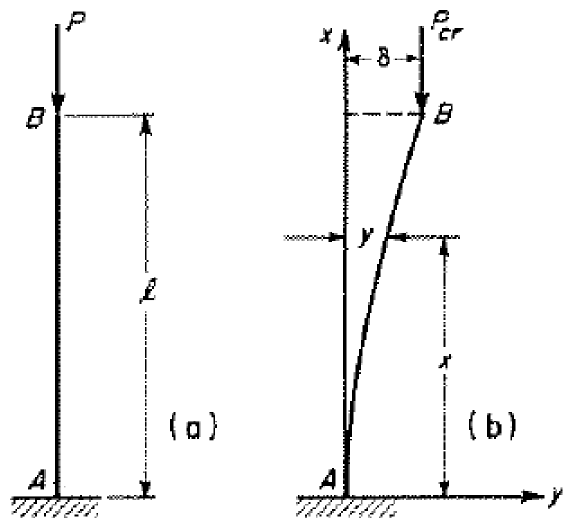
14.1 Introduction:

Structural engineers commonly analyze structures using “first order” finite element analysis packages. In so doing, we make the assumption that as a structure deflects, changes in the structure’s geometry will have little or no effect on the structure’s rigidity. For the majority of structural analysis problems, first order theory is sufficiently accurate. However, we need to recognize certain situations where we can expect large changes in geometry which may lead to overall instability.

This chapter is an overview of structural analysis methods used to for non-linear type problems. We begin with a simple column non-linear analysis; then we will extend similar principles to non-linear frame analysis. Finally, we will describe two of DCALC’s specialized programs written for non-linear analysis.

14.2 A Classic Non-Linear Problem: Perfectly Straight Columns

A classic example of non-linear behavior comes from elementary column theory.



The differential equation for a straight column with axial load, “P” is,

$$Moment = EI * \frac{d^2y}{dx^2} = P * (\delta - y)$$

Therefore, the column moment is effected by the geometry, “y” and by the column load “P”.   We commonly refer to this moment as the “P*Δ” effect.

*(http://www.designcalcs.com)

This leads to a somewhat perplexing situation: If we start with a straight column, "y" is zero. But if we move the column laterally by even the smallest distance we can expect lateral deflection "y" to grow. Below a critical load, call it "Pcr", if we wiggle the column laterally, it will snap back to vertical. But at "Pcr", if we wiggle the column laterally, "y" will increase until the column buckles.

Therefore, for a perfectly straight column there are only two conditions: straight or buckled. It's "all or nothing" (or more appropriately, "nothing or all".) There is no "in between" condition where the column will remain in a distinct shape if we push it laterally.

The mathematician Euler was able to solve the column differential equation, which leads to his famous column buckling equation:

$$Pcr = \pi^2 EI/(4l^2)$$

14.3 Less-Than-Straight Columns

In the early part of the twentieth century, engineers realized that the Euler equation predicted column loads that were much too high for real world columns which are less-than-straight. Engineers sought to develop a rational basis for safety factors used to design columns. Simply "dividing the Euler critical load by two" was no guarantee that a column would be safe. This topic was of particular interest to Professor Stephen Timoshenko.

Perhaps no person has had a greater influence on structural engineering education – specifically ***engineering mechanics*** - than Stephen Timoshenko.

Born and educated in Ukrainia, Timoshenko taught at the University of Michigan during the 1920's and 1930's, where he created the first bachelor's and doctoral programs in engineering mechanics. Timoshenko later taught at Stanford.

Many of Timoshenko's writings were concerned with instability problems, such as the buckling of columns, plates and beams. His textbooks organized engineering mechanics topics, essentially pioneering the discipline as a subject which can be taught.

Stephen Timoshenko
(1878-1972)

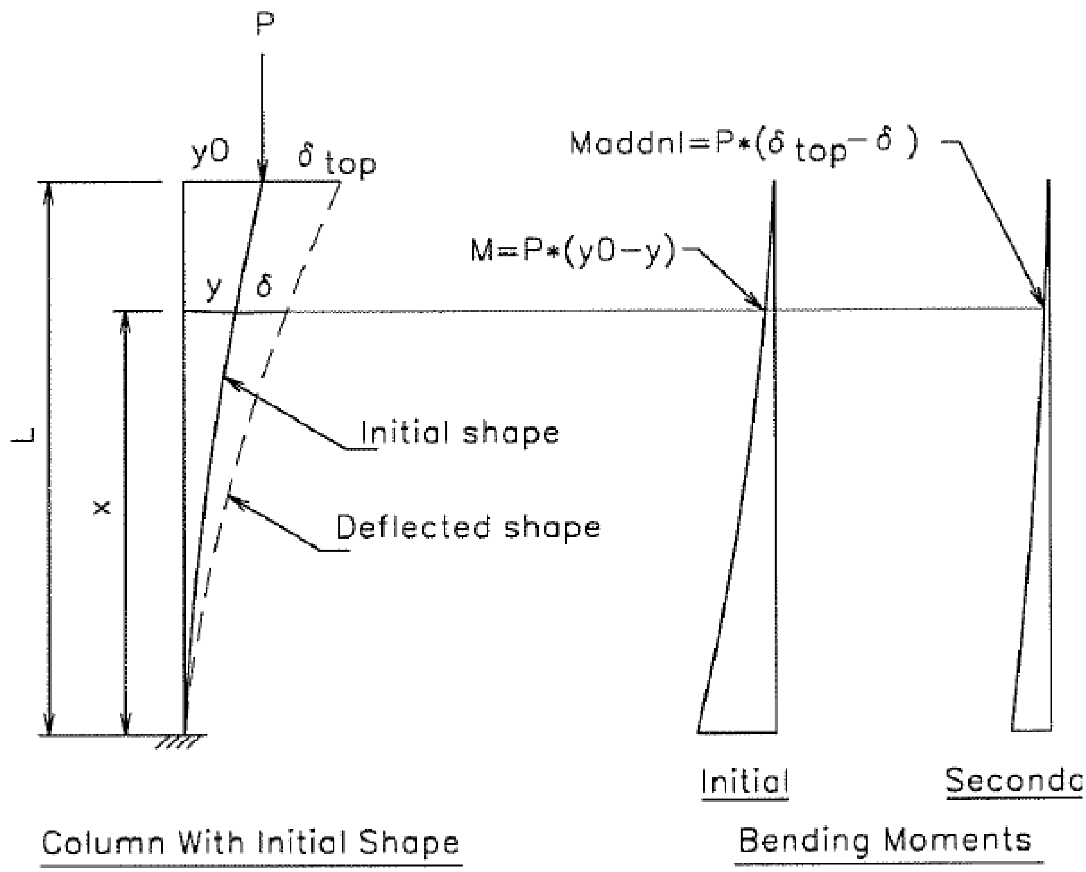
In the days before computers, engineering theorists solved closed formed solutions to differential equations for all types of elementary non-linear buckling problems. For example, solutions to column stability, lateral torsional buckling and web plate shear buckling stability are the very foundation of modern steel design specifications.

14.4 Solving the Colmun Problem Iteratively

Soon engineering theorists recognized that column load analysis could be approached much more easily by assuming an initial shape, such as a sine wave or a parabola. \Various approaches where developed using this technique, such as:

- Timoshenko described "The Method of Successive Approximations in Calculations Critical Loads". In this method he computed the lateral deflection in steps. (Ref. 1)
- "Raleigh's method" is a similar column buckling load prediction method. Instead of computing deflections, this method computes energy (Ref. 2)

Today we call such techniques "iterative" numerical methods. The below example will illustrate one such algorithm that can be used to compute column buckling loads:



We can compute the lateral deflection of a column with an initial shape as follows:

1. Assume an initial deflection. The maximum amplitude may be set to a normal inspection tolerance; for example, L/400 in the case of columns. Therefore, in the above figure, we might assume the shape to be,
$$y = y0 * \left(1 - \cos\left(\frac{\pi}{2} * \frac{x}{L}\right)\right)$$
where y0=L/400
2. Compute initial moments along the column at a number of point: M=P*(y0-y)

3. Compute (*additional*) deflections, δ, due to moment, M. This can be done easily in a spreadsheet using the conjugate beam method, or by using a general structural analysis software

4. Adjust the geometry "y" ($y = y_{previous} + \delta_{additional}$)

5. Compute ***additional secondary*** moments: $M_{addn'l} = P*(\delta 0 - \delta)$

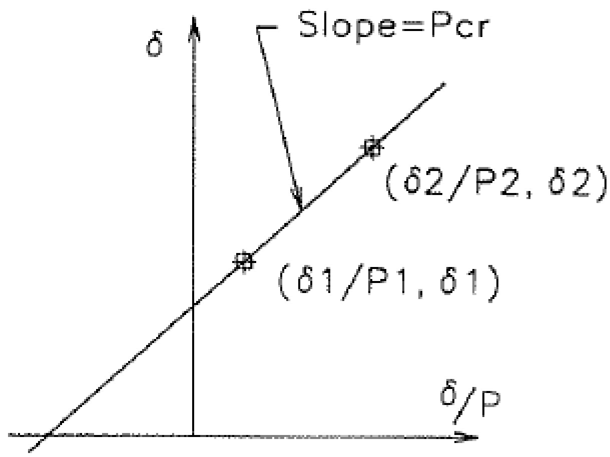
6. Goto Step 3 (compute ***additional*** δ due to $M_{addn'l}$)

The above steps are repeated until the shape converges into a final shape, having a total deflection "δ" at the top. (Of course if "P" exceeds Pcr, the shape will not converge.)

The buckling load can then be predicted using a "Southwell Plot", as follows:

1. Compute the column shape for two loading cases, P1 and P2, resulting in two deflections, δ1 and δ2 at the top of the column.

2. Plot results in a "Southwell Plot" (Ref. 1)



Southwell Plot

3. **The slope of the line is the critical buckling load, Pcr.**

Interestingly, this method using a Southwell Plot can be generalized and can be applied to such problems as beam, truss and arch buckling.

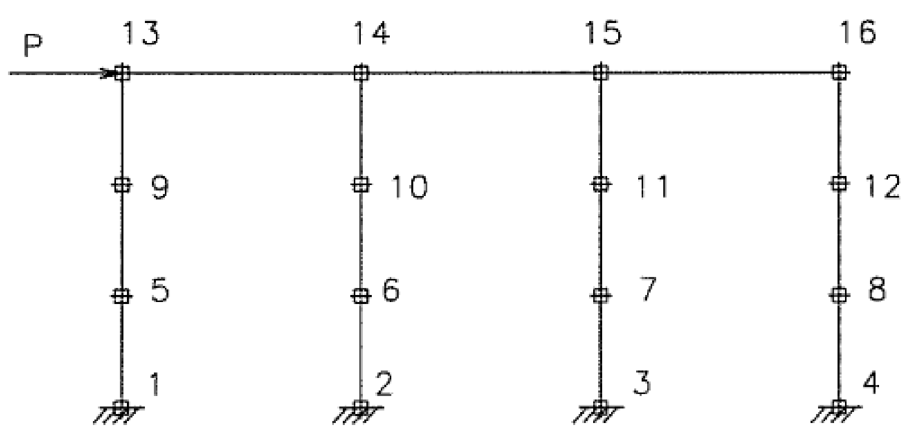
14.5 Non-Linear Frame Analysis

Today we routinely analyze structures using general structural analysis software. Virtually all general structural analysis software is based on the finite element stiffness method (FEM). We can expect that structural analysis results for a frame will be virtually the same for all these different software packages that use first order analysis.

When we get into non-linear frame analysis, there are several different approaches that can be used, each of which may produce different results. The following is a list of various approaches that could be used (Ref. 3):

- First order elastic
- Second order elastic: In this method, changes to the frame geometry are considered for each iteration
- First order elastic plastic: In this method, changes to the frame geometry are not considered, but plastic hinges are introduced as required
- Second order elastic plastic: In this method, changes to the frame geometry are considered for each iteration, and “simple” plastic hinges are introduced
- Second order spread of plasticity: In this method, changes to the frame geometry are considered for each iteration, and more realistic plastic hinges are introduced which may spread over a plastic zone. (Most realistic analysis)

We will be discussing “Second order elastic plastic” analysis using the Newton-Raphson Method (Ref. 3). This method is quite similar to the previously described column problem, but with important differences. In lieu of mathematical details, we will demonstrate this method using the following example.



Suppose we want to compute deflections for the above frame by using a second order analysis. Let’s say we want to include plastic hinging effects when column moments reach their plastic moment capacity. Also, we want to predict the lateral “P” load that will push (collapse) the frame over. Hence, we expect to see a sort of “push-over” analysis in our results.

The following algorithm describes the steps involved:

1. Determine joints numbers and members.

2. Define initial coordinates (x,y,z) for each joint

3. Determine member section properties (A, Ix, Iy, Iz) and modulus of elasticity (E)

4. Determine the plastic capacity at each end of the member. Ideally, a failure enveloped can be defined (an interaction curve), showing “P versus M”. A simpler alternative, although less realistic, is to compute a plastic moment, “Mp”, that is not influenced by axial load, “P”.

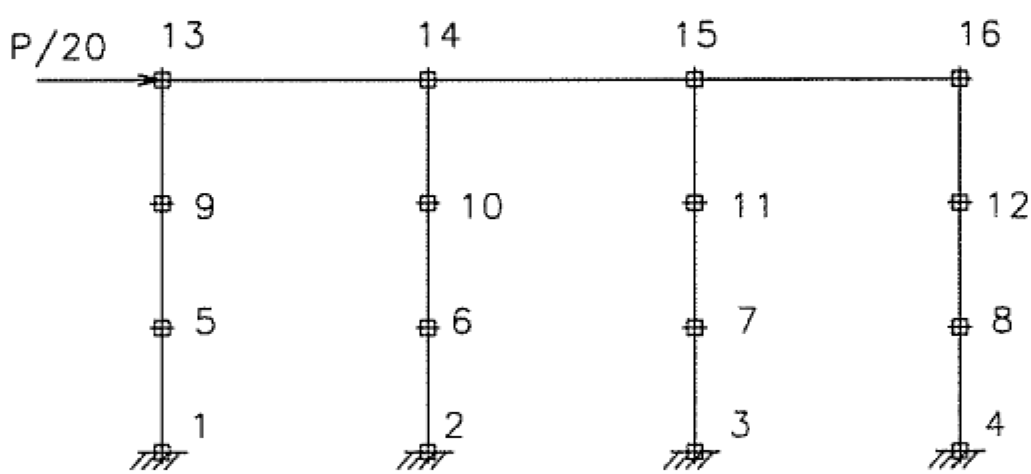
5. Define one loading case to be applied to the structure. In this example, only one load, “P” will be considered.

We will be solving this problem iteratively using 20 iterations.

6. Apply P/20 to the frame using the initial geometry.

7. Compute deflections and internal forces in all the members due to P/20

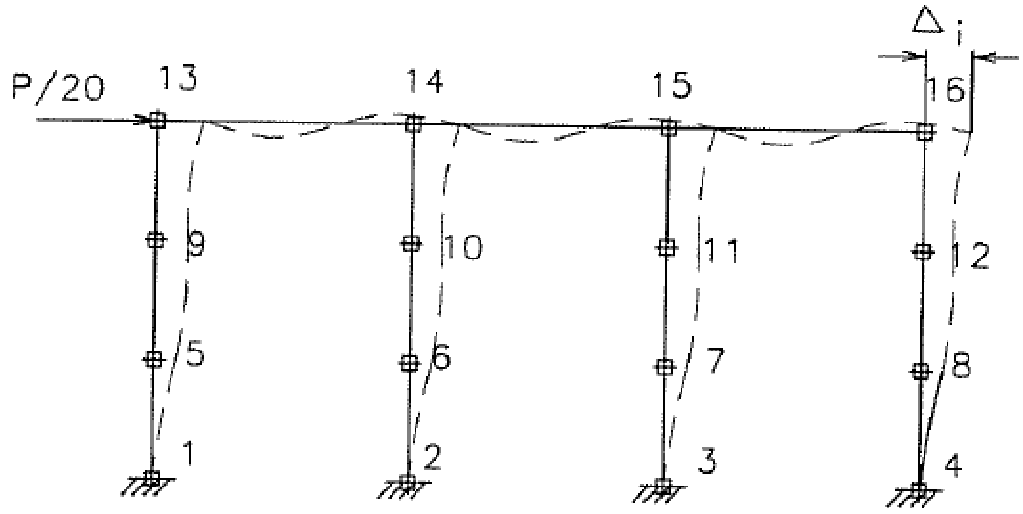
8. Store these deflections and internal forces in “total deflection” and “total internal force” arrays



Initial Application of Load

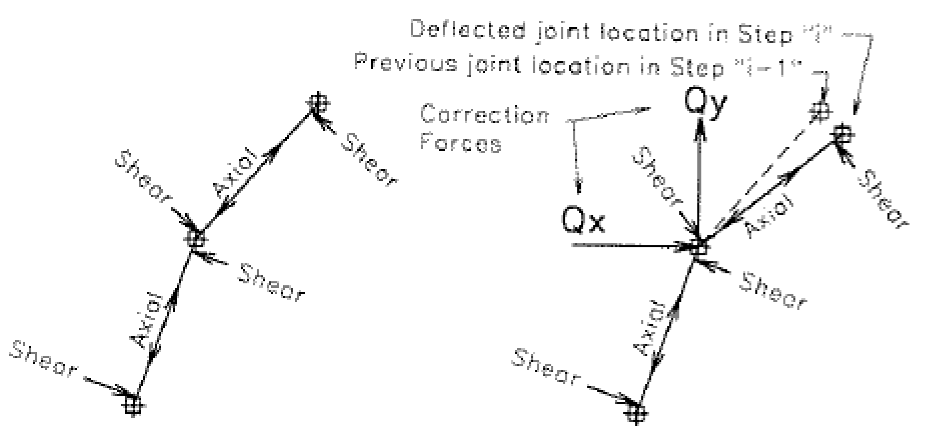
9. For the next step, **adjust all joint locations, moving the joint locations by the deflections computed in the previous step**.

10. Compute an updated (softer) stiffness matrix, based on the updated joint locations
11. For next 19 subsequent steps "i", we will compute the incremental deflections in two parts:
    a. Compute the deflection, Δ_i, due to P/20:



Step "i": Application of Incremental Load

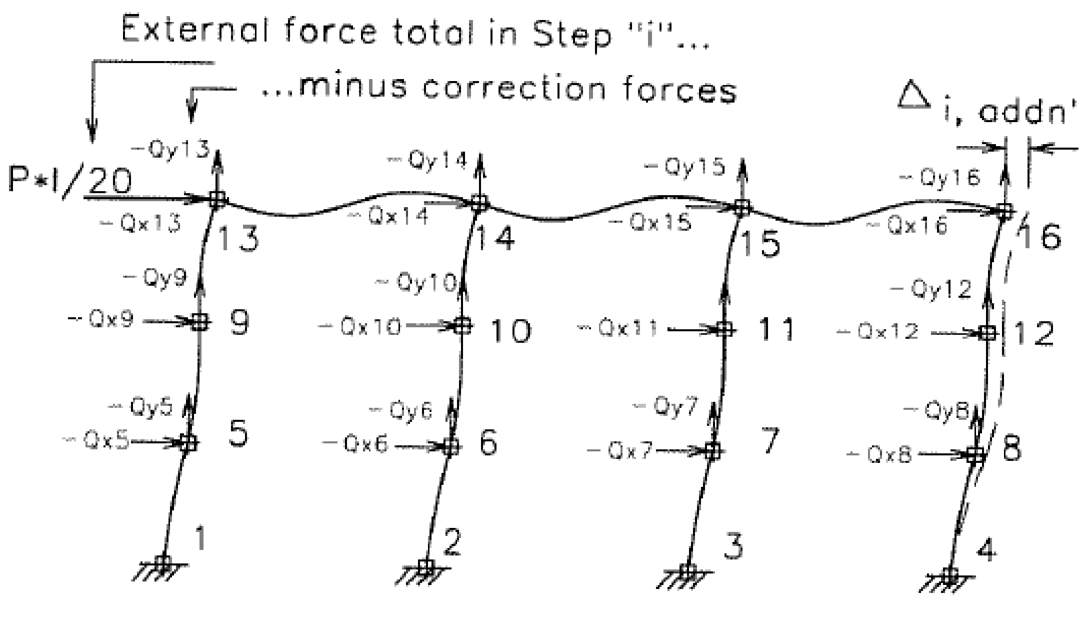
    b. Update the geometry and the stiffness matrix
    c. **Compute the correction forces,. "Q", at each joint required to maintain force equilibrium using the current total internal forces**:



Internal Member Forces In Step "i-1"

Joint Correction Forces In Step "i"

d. Compute the additional displacement, $\Delta_{i,addn'l}$, by applying the <u>difference</u> between the current <u>total external loads</u> and the correction "Q" forces:



Step "i": Computation of Additional Deflection

12. Add the deflections, Δ_i and $\Delta_{i,addn'l}$, to the "total displacement" array and member forces to "total internal force" array.

13. Check if total member forces at each member end exceed the plastic force limit. **If the plastic limit is reached, the member end fixity is revised to a hinge for all subsequent loadings**.

14. Go to Step 10 until all iterations (20 in this example) *or until the structure becomes a mechanism*. (A mechanism will be signaled when the structure stiffness matrix cannot be inverted)

Using the above procedure, a complete history of deflections, including when plastic hinges are formed, can be developed. Using animation techniques, we can view how the frame deforms, and when plastic hinging form, and how a structure will collapse. Using this method, we are closer to understanding true frame behavior.

14.6 Specialized Non-Linear Analysis Software Applications

It would be a tedious task to attempt making a non-linear analysis using only a first order structural analysis package and a spreadsheet. For routine non-linear analysis, specialized software is required. This variety of software is essentially the same as standard first-order finite element software, but has the additional iterative steps, such as described above.

DCALC presently has two non-linear analysis program:

- **UNLINEAR**: This program performs a non-linear analysis of a frame that has been input in the DCALC program "FRAME".
- **PILEBUCL**: This program analyzes an end bearing pile for lateral stability

Both of these programs use the Newton-Raphson Methodology described above.

14.7 UNLINEAR Program

DCALC's "UNLINEAR" program performs a second-order elastic plastic analysis using the Newton-Raphson Methodology, as described in Section 14.5 The designer will go through the following steps:

1. Model the structure in the FRAME program

2. In FRAME, apply one load case which will subsequently be used for the non-linear analysis

3. Before using UNLINEAR, compute the plastic moments of members. If you are expecting plastic hinging to occur at certain points in the structure, compute the plastic moment at the hinges. UNLINEAR uses a "simple plastic" model, where a hinge is introduced when the moment reaches the plastic moment.

4. Run UNLINEAR.
    - You will be asked to retrieve the FRAME file
    - You will be asked to identify the load case
    - You will be asked to input plastic moments at the ends of all the members (Sometimes it's easiest to copy the same capacity to all members, if you know where to expect plastic hinging will occur.)

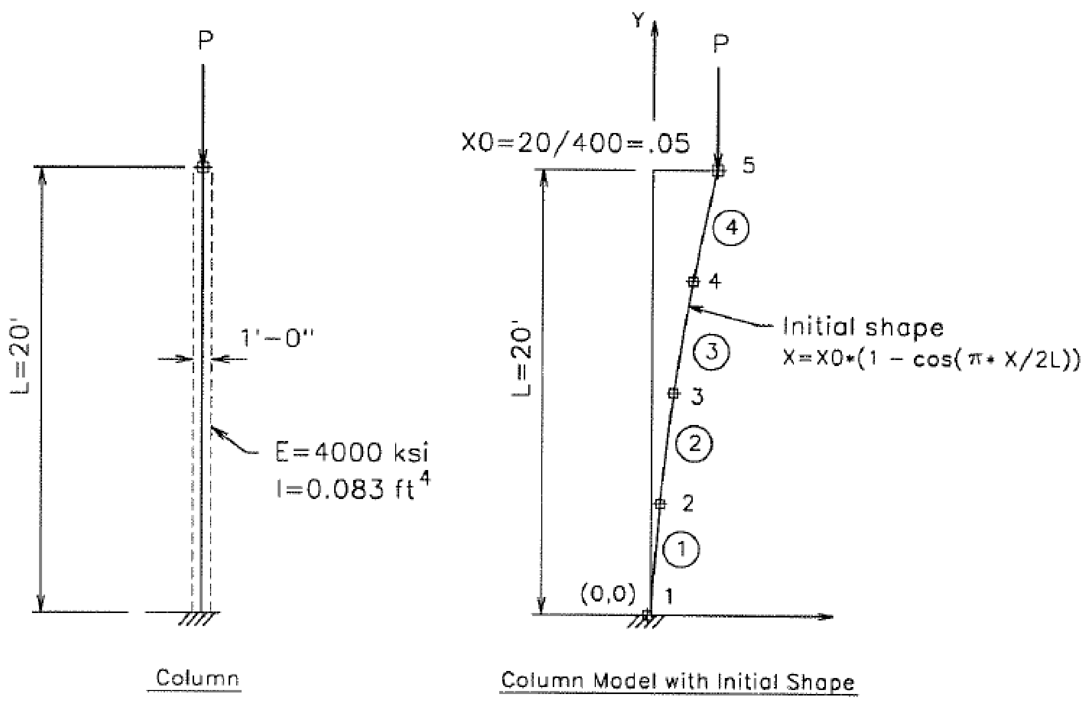
5. Select "Perform Non-Linear Analysis and Save Calcs"

6. Select "View Deflected Structure"

The above steps are actually very simple. The construction of the model in FRAME is the majority of the work required.

We will be showing how UNLINEAR can be used in the following example problem.

14.8 Example Problem: Solving a Buckling Problem Using A Southwell Plot
The following is a simple example showing how to use UNLINEAR and predict a buckling load using a Southwell Plot.



The buckling load for a perfectly straight column is computed by the Euler equation:

$$Pcr = \pi^2 EI/(4l^2)$$
$$= \pi^2 * \frac{(4000 * 144)(0.083)}{4 * 20^2}$$
$$= 294.9\ k$$

We will next predict the buckling load using UNLINEAR and a Southwell Plot:

1. Run FRAME. Input the model shown above right, fitting the column with an initial sine wave imperfection, with a tolerance of X0=20/400 = .05 ft at the top

2. We enter two loading cases, less than the buckling load:
   Load 1: P1 = 100 k at Joint 5
   Load 2: P2 = 200 k at Joint 5

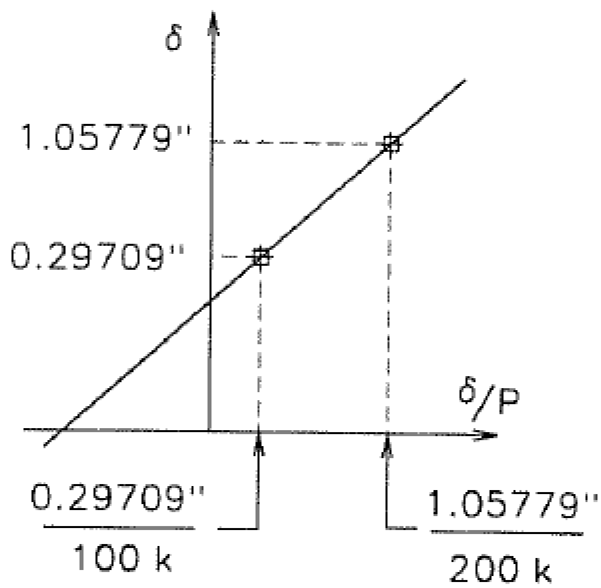
3. We then save the FRAME input file

4. Next run UNLINEAR.

5. UNLINEAR computes the following results:

   For Load 1 with P1=100 k, Δx1 = 0.29709inch
   For Load 2 with P2=200 k, Δx2 = 1.05779 inch

6. These values are plotted in a Southwell Plot



Southwell Plot

7. Compute the critical load as the slope of the line:

$$Pcr = \frac{1.05779 - .29709}{\frac{1.05779}{200} - \frac{0.29709}{100}}$$

$$= 328\,k \quad > 298\,k$$

Although the predicted buckling load is greater than the Euler load, it is fairly close.

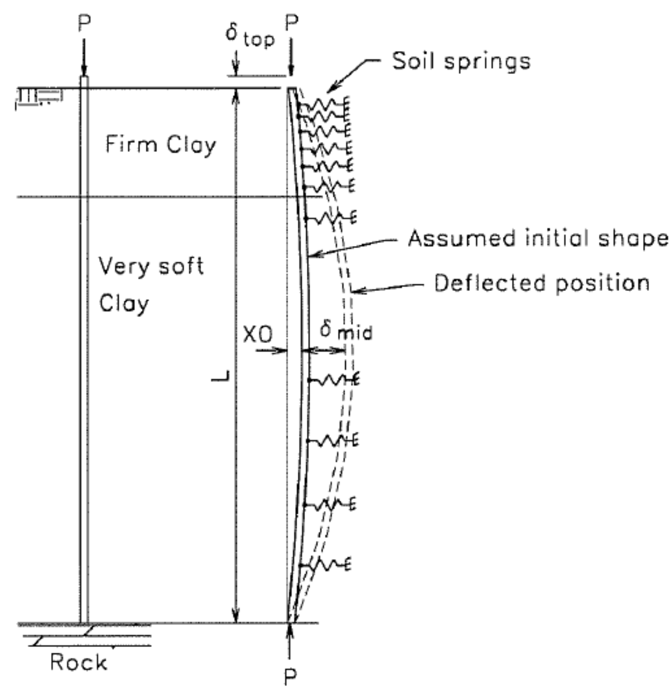
This example demonstrates that this approach, as implemented here in UNLINEAR, tends to model structures as slightly stiffer than the actual structure. There are algorithms which use more steps involving the correction forces, which will converge even closer to the exact solution. In principle, the tendency is to under-predict deflections using this method.

14.9 PILEBUCL Program

Typically piles are designed without much concern about lateral stability. Usually the surrounding soil is sufficiently stiff to brace a pile from buckling. Traditionally steel end bearing piles are designed for an end bearing stress, regardless of the length of the pile.

There are rare occasion when a designer is faced with the using very long end bearing piles driven through very soft soil. For situations such as these there may be a concern that about pile's lateral stability.

DCALC's PILEBUCL program was written as a way to evaluate "what if" scenarios. The below sketch shows the basic setup:



The designer must first assume that the pile is driven in an initial shape, with an ordinate of "X0" at the midpoint. The designer must exercise some judgment about installation tolerances (*). PILEBUCL automatically generates a 51 node structural model, fitting a sine wave through "X0". Pile soil springs are automatically generated based on Bowles soil spring recommendations (Ref. 4). A Newton-Raphson non-linear analysis is performed, as outlined in the previous section. The designer can plot mid-depth displacements, "δmid", in a Southwell Plot, to predict the pile buckling load.

(*) Obviously, it's impossible to determine the initial shape of a pile underground. The designer must exercise some judgment about how the pile will be installed. Therefore this analysis can be an "evaluation" at best. If the situation appears particularly sensitive, it may be prudent to require pile load tests. It would appear that in principle, a Southwell Plot of recorded top displacements,"δtop", can be made to predict the pile buckling load from full scale load tests.

List of References:

Reference 1: "Theory of Elastic Stability", S. Timoshenko, published by The Maple Press Company, 1936

Reference 2: "Advanced Strength of Materials", J. P. Den Hartog, Dover Publications, 1952

Reference 3: "Stability Design of Steel Frames", W. F. Chen and E. M. Lui, published by CRC Press, 1991

Reference 4: "Foundation Analysis and Design", 4th Edition, J. E. Bowles, published by McGraw Hill, 1988

Other Resources:

Websites:

- Image of Timoshenko taken from Wikipedia